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The second Josephson harmonic in the dirty limit

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Abstract. We calculate the Josephson current across a plane interface separating dirty superconductors up to second order in the transparency of the junction. The leading contribution to the second-order correction is caused by the suppression of the superconducting order parameter near the interface. The ratio of the correction to the Josephson current in the first order is about $D\xi(T)/l$, where D is the transparency of the junction, $\xi(T)$ is the temperature-dependent coherence length, and l is the mean free path. This correction gives rise to the second harmonic in the current–phase relationship.

1. Introduction

The charge transport in mesoscopic superconducting–normal-metal hybrid structures has become a field of intensive theoretical and experimental research (for a recent review see reference [1]). A useful theoretical tool for studying equilibrium and nonequilibrium properties of these structures is provided by the quasiclassical equations of superconductivity (see e.g. reference [1, 2]). The realistic mesoscopic structures often turn out to be in a dirty limit, when the mean free path l is much shorter than the zero-temperature coherence length ξ_0 in a superconductor. Quasiclassical equations in this case simplify to the Usadel limit, which is characterized by keeping only the first two terms in the expansion of the angular dependence of the quasiclassical propagators in Legendre polynomials $P_n(\mu)$.

The quasiclassical equations of superconductivity must be supplemented by the boundary conditions, which were derived by Zaitsev [3] in a general form valid in both the clean and dirty limits. Kuprianov and Lukichev [4] considered the problem of writing Zaitsev's boundary condition in terms of the Usadel form of the quasiclassical propagator. However, as was demonstrated in reference [5] the boundary condition of Kuprianov and Lukichev is valid only to the lowest order in the expansion in the small parameter D (transparency of the barrier). In the general case there appear higher terms in an expansion over Legendre polynomials $P_n(\mu)$. All higher terms ($n \geqslant 2$) decay exponentially with the distance from the interface (the characteristic scale is l). Nevertheless, they essentially influence the boundary condition. Lambert *et al* considered the problem up to the order D^2. When they treated the problem of the Josephson current, they obtained the second harmonic of the current–phase relationship in this order. But Lambert *et al* did not consider the change of the superconducting order parameter near the junction, which (as we shall show) takes place in the first order in D. When this suppression is taken into account, the magnitude of the second harmonic is increased by $\xi(T)/l$ (with $\xi(T)$ standing for the temperature-dependent coherence length).

Our paper is organized as follows. In section 2 we briefly review the Eilenberger equations and Zaitsev's boundary conditions. In section 3 we suggest a convenient real-vector form of the Eilenberger equations. Then in section 4 we find the perturbational expressions for the quasiclassical Green's function in the first order in transparency. First we employ an approximation allowing us to describe the behaviour of the superconducting order parameter on the scale of the order l near the junction. This provides a boundary condition at the interface for the variation of the order parameter on a generic scale $\xi(T)$. For the description of the latter one can use the Usadel equation. So we are in a position to estimate the superconducting order parameter near the interface and to write down the result for the Josephson current in the second order in transparency.

In this paper we discuss the stationary Josephson effect. The pairing is assumed to be singlet and isotropic (the elastic scattering by nonmagnetic impurities is detrimental to anisotropic superconductivity and only the s-wave type of pairing can exist in the dirty limit). For simplicity we are considering the contact of identical superconductors.

2. The Eilenberger equations and Zaitsev's boundary conditions

The main building block in the quasiclassical theory of equilibrium superconductivity is a propagator which has a 2 × 2 matrix structure:

$$\hat{g}(\boldsymbol{p}, \boldsymbol{R}, \omega_m) = \begin{pmatrix} g(\boldsymbol{p}, \boldsymbol{R}, \omega_m) & \mathrm{i}f(\boldsymbol{p}, \boldsymbol{R}, \omega_m) \\ -\mathrm{i}f^+(\boldsymbol{p}, \boldsymbol{R}, \omega_m) & -g(\boldsymbol{p}, \boldsymbol{R}, \omega_m) \end{pmatrix}. \tag{1}$$

Here $\boldsymbol{p}$ denotes position on the Fermi surface, $\boldsymbol{R}$ stands for position in space, and $\omega_m = (2m+1)\pi T$ is the Matsubara frequency. The quasiclassical propagator has a normalization property $\hat{g}^2 = 1$, resulting in the additional relationship $g^2 + ff^+ = 1$. Anomalous (f, f^+) and normal (g) Green's functions have important symmetry relationships:

$$\begin{aligned} f^{+*}(\boldsymbol{p}, \boldsymbol{R}, \omega_m) &= f(-\boldsymbol{p}, \boldsymbol{R}, \omega_m) = f(\boldsymbol{p}, \boldsymbol{R}, -\omega_m) \\ g^*(\boldsymbol{p}, \boldsymbol{R}, \omega_m) &= g(-\boldsymbol{p}, \boldsymbol{R}, \omega_m) = -g(\boldsymbol{p}, \boldsymbol{R}, -\omega_m). \end{aligned} \tag{2}$$

The Eilenberger equations can be written in a concise matrix form as

$$\mathrm{i}\boldsymbol{v}_F \cdot \boldsymbol{\nabla}\hat{g} + \hat{\omega}\hat{g} - \hat{g}\hat{\omega} = 0 \tag{3}$$

where

$$\hat{\omega} = \left(\mathrm{i}\omega_m + \frac{e}{c}\boldsymbol{v}_F \cdot \boldsymbol{A}\right)\hat{\sigma}_z - \hat{\Delta} + \frac{\mathrm{i}}{2\tau}\langle\hat{g}\rangle + \frac{\mathrm{i}}{2\tau_s}\langle\hat{\sigma}_z\hat{g}\hat{\sigma}_z\rangle$$

where: $\boldsymbol{v}_F(\boldsymbol{p})$ is the Fermi velocity; $\boldsymbol{A}$ stands for the vector potential; τ and τ_s are the times for elastic scattering on nonmagnetic and paramagnetic impurities, respectively. The angular brackets denote averaging over the Fermi surface. The matrix $\hat{\Delta}(\boldsymbol{R})$ incorporates the superconducting order parameter $\Delta(\boldsymbol{R})$; $\hat{\sigma}_z$ is the Pauli matrix:

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The superconducting order parameter (otherwise called the pairing potential) and the current density $\boldsymbol{j}(\boldsymbol{R})$ are given by

$$\begin{aligned} \Delta(\boldsymbol{R}) &= \pi T\lambda \sum_m \langle f(\boldsymbol{p}, \boldsymbol{R}, \omega_m)\rangle \\ \boldsymbol{j}(\boldsymbol{R}) &= -2\pi\mathrm{i}eTN(0)\sum_m \langle \boldsymbol{v}_F(\boldsymbol{p})g(\boldsymbol{p}, \boldsymbol{R}, \omega_m)\rangle. \end{aligned} \tag{5}$$

$N(0)$ is the density of states at the Fermi energy per spin direction and λ is the superconducting coupling constant. We shall denote the phase of complex $\Delta(\boldsymbol{R})$ as $\chi(\boldsymbol{R})$. We shall set the vector potential to be zero. The latter circumstance does not rule out the possibility of studying current-carrying states (the Josephson current) in our formalism, since the superfluid velocity can be created by the gradient of the phase.

Zaitsev's boundary conditions [3] in the matrix notation can be written as

$$\begin{aligned} &\hat{g}_{a+} = \hat{g}_{a-} = \hat{g}_a \\ &\hat{g}_a\big[(1 - D(\boldsymbol{p}))(\hat{g}_{s+} + \hat{g}_{s-})^2 + (\hat{g}_{s+} - \hat{g}_{s-})^2\big] = D(\boldsymbol{p})\big[\hat{g}_{s+}\hat{g}_{s-} - \hat{g}_{s-}\hat{g}_{s+}\big]. \end{aligned} \tag{6}$$

Here

$$\hat{g}_{s,a}(\boldsymbol{p}, \boldsymbol{R}, \omega_m) = [\hat{g}(\boldsymbol{p}, \boldsymbol{R}, \omega_m) \pm \hat{g}(\boldsymbol{p}_r, \boldsymbol{R}, \omega_m)]/2$$

(by $\boldsymbol{p}_r$ we denote the reflected momentum). The subscripts $\pm$ in equation (6) stand for the expressions that are taken on the right-hand (left-hand) side of the interface, respectively. Finally, $D(\boldsymbol{p})$ is the transparency coefficient of the boundary for the electron at the Fermi surface with the given direction of momentum. The first of these relationships results in the current conservation at the boundary. When $D(\boldsymbol{p})$ is equal to unity, the boundary conditions give rise to continuous $\hat{g}$.

3. The real-vector parametrization of Eilenberger equations

The transparency coefficient for the case under discussion depends on the angle between the Fermi velocity and the normal to the boundary. In the model of the δ-like boundary potential it is given by

$$D(\boldsymbol{p}) = \frac{p_x^2}{U_0^2 + p_x^2} \tag{7}$$

with parameter U_0 accounting for the strength of the potential. The interface is assumed to be perpendicular to the x-axis (the junction is supposed to be located at $x = 0$). Hence the quasiclassical propagator depends on p_x, x, and ω_m. Let us introduce the functions

$$g_{s,a}(p_x, x, \omega_m) = [g(p_x, x, \omega_m) \pm g(-p_x, x, \omega_m)]/2$$

and similarly defined functions $f_{s,a}$, $f^+_{s,a}$. Due to the symmetry relationships (2) these can be parametrized as

$$\begin{aligned} &g_s = b_{s1} \qquad f_s = b_{s2} - \mathrm{i}b_{s3} \qquad f_s^+ = b_{s2} + \mathrm{i}b_{s3} \\ &g_a = \mathrm{i}b_{a1} \qquad f_a = b_{a3} + \mathrm{i}b_{a2} \qquad f_a^+ = -b_{a3} + \mathrm{i}b_{a2}. \end{aligned} \tag{8}$$

The parameters b_s, b_a are real and are combined into two three-dimensional vectors $\boldsymbol{b}_s = (b_{s1}, b_{s2}, b_{s3})$ and similarly for $\boldsymbol{b}_a$. By forming sums and differences of the Eilenberger equations for $\hat{g}(\pm p_x, x, \omega_m)$, one can arrive at equations for $\boldsymbol{b}_{s,a}$:

$$\frac{\mathrm{d}\boldsymbol{b}_s}{\mathrm{d}x} = \boldsymbol{M} \times \boldsymbol{b}_a \qquad \frac{\mathrm{d}\boldsymbol{b}_a}{\mathrm{d}x} = -\boldsymbol{M} \times \boldsymbol{b}_s. \tag{9}$$

On the right-hand side of these equations we have the vector product of three-dimensional vectors. The arguments p_x, x, and ω_m of $\boldsymbol{b}_{s,a}$ were omitted for brevity. The equations in (9) should be considered only for $p_x > 0$ and $\omega_m > 0$ (this is the elimination of the redundancy mentioned above). The three-dimensional vector $\boldsymbol{M}$ in (9) is real. Its components are given by

$$\begin{aligned} &M_1 = 2\tilde{\omega}/v_x \qquad M_2 = (\tilde{\Delta} + \tilde{\Delta}^*)/v_x \qquad M_3 = \mathrm{i}(\tilde{\Delta} - \tilde{\Delta}^*)/v_x \\ &\tilde{\omega}(x, \omega_m) = \omega_m + \left(\frac{1}{2\tau} + \frac{1}{2\tau_s}\right)\langle g(p_x, x, \omega_m)\rangle \\ &\tilde{\Delta}(x, \omega_m) = \Delta(x) + \left(\frac{1}{2\tau} - \frac{1}{2\tau_s}\right)\langle f(p_x, x, \omega_m)\rangle. \end{aligned} \tag{10}$$

v_x stands for the x-component of Fermi velocity $\boldsymbol{v}_F(\boldsymbol{p})$. The components of $\boldsymbol{M}$ (and the pairing potential) can also be expressed via $\boldsymbol{b}_s$:

$$\begin{aligned} v_x M_1 &= 2\omega_m + \left(\frac{1}{\tau} + \frac{1}{\tau_s}\right)\langle b_{s1}\rangle \\ v_x M_{2,3} &= 4\pi T\lambda \sum_{\omega_m>0} \langle b_{s2,3}\rangle + \left(\frac{1}{\tau} - \frac{1}{\tau_s}\right)\langle b_{s2,3}\rangle. \end{aligned} \tag{11}$$

The electric current is related to b_{a1} by

$$j = 4\pi e T N(0) \sum_{\omega_m>0} \langle v_x b_{a1}(p_x, x, \omega_m)\rangle. \tag{12}$$

The angular averaging in equations (11), (12) is carried out for $p_x > 0$. Zaitsev's boundary conditions result in

$$\begin{aligned} &\boldsymbol{b}_{a-} = \boldsymbol{b}_{a+} = \boldsymbol{b}_a \\ &\left[(\boldsymbol{b}_{s+} - \boldsymbol{b}_{s-})^2 + (1-D)(\boldsymbol{b}_{s+} + \boldsymbol{b}_{s-})^2\right]\boldsymbol{b}_a = 2D\boldsymbol{b}_{s+} \times \boldsymbol{b}_{s-}. \end{aligned} \tag{13}$$

As a consequence of the normalization condition we obtain

$$\begin{aligned} &g_s^2 + f_s^+ f_s + g_a^2 + f_a^+ f_a = 1 \\ &f_s^+ f_a + f_a^+ f_s + 2g_a g_s = 0 \end{aligned} \tag{14}$$

or equivalently

$$\boldsymbol{b}_s^2 = 1 + \boldsymbol{b}_a^2 \qquad \boldsymbol{b}_s \boldsymbol{b}_a = 0. \tag{15}$$

Multiplying the first of equations (9) by $\boldsymbol{b}_s$ and subtracting the second equation multiplied by $\boldsymbol{b}_a$ we get $\boldsymbol{b}_s^2 - \boldsymbol{b}_a^2 = \text{constant}$. Multiplying the first of equations (9) by $\boldsymbol{b}_a$ and adding the second equation multiplied by $\boldsymbol{b}_s$ we get $\boldsymbol{b}_s\boldsymbol{b}_a = \text{constant}$. So the normalization conditions (15) are consistent with the equations for $\boldsymbol{b}_{s,a}$.

It is useful to note that if we multiply the equation for $\mathrm{d}b_{a1}/\mathrm{d}x$ by v_x and integrate the resulting expression over the Fermi surface and sum over Matsubara frequencies, we obtain the current conservation $\mathrm{d}j_x/\mathrm{d}x = 0$. It is necessary in this derivation to use expression (5) for the pairing potential.

4. The Josephson current correction

In this section we shall find the vectors $\boldsymbol{b}_s, \boldsymbol{b}_a$ up to the first order in transparency. It is convenient to represent

$$\begin{aligned} &\boldsymbol{b}_s = \boldsymbol{b}_s^{(3)} + \delta\boldsymbol{b}_s \\ &\boldsymbol{b}_s^{(3)} = \frac{1}{\sqrt{\omega_m^2 + |\Delta(x)|^2}} \begin{pmatrix} \omega_m \\ |\Delta(x)|\cos\chi(x) \\ -|\Delta(x)|\sin\chi(x) \end{pmatrix}. \end{aligned} \tag{16}$$

Note that the vector $\boldsymbol{b}_s^{(3)}$ incorporates corrections in the transparency D because of the D-dependence of $|\Delta(x)|$ and $\chi(x)$. This vector gives the solution of the Eilenberger equations for the bulk, when the pairing potential is constant and magnetic impurities are absent (for the case of the s-wave pairing considered in this paper this is true irrespective of the value of $1/\tau$, but the situation changes when the pairing is anisotropic). The square of the vector $\boldsymbol{b}_s^{(3)}$ is unity, and from the normalization conditions (15) we see that in the first order in D

$$\boldsymbol{b}_s^{(3)}\,\delta\boldsymbol{b}_s = 0 \qquad \boldsymbol{b}_s^{(3)}\boldsymbol{b}_a = 0 \tag{17}$$

so we can write

$$\delta \boldsymbol{b}_s = C_{s1}\boldsymbol{e}_1 + C_{s2}\boldsymbol{e}_2 \qquad \boldsymbol{b}_a = C_{a1}\boldsymbol{e}_1 + C_{a2}\boldsymbol{e}_2$$
$$\boldsymbol{e}_1 = \begin{pmatrix} 0 \\ \sin\chi(x) \\ \cos\chi(x) \end{pmatrix} \tag{18}$$
$$\boldsymbol{e}_2 = \frac{1}{\sqrt{\omega_m^2 + |\Delta(x)|^2}} \begin{pmatrix} |\Delta(x)| \\ -\omega_m \cos\chi(x) \\ \omega_m \sin\chi(x) \end{pmatrix}.$$

We shall write down the superconducting phase as $\chi(x) = (\chi_0/2) + \delta\chi(x)$ for $x > 0$ and as $\chi(x) = -(\chi_0/2) - \delta\chi(x)$ for $x < 0$. The function $\delta\chi(0) = 0$, so there is a phase jump χ_0 at the junction.

An essential approximation, which is justified in the dirty limit, is to take $M_i = \langle b_{si}\rangle/\tau v_x$ (the influence of paramagnetic impurities will be neglected). It turns out that there is a variation of the Green's functions (linear in D) on the scale l near the junction, so this approximation is valid within distances of the order of l near the junction. The omitted terms are small in the measure of l/ξ_0. Writing down equation (9), we obtain two independent systems

$$\frac{\mathrm{d}C_{s2}}{\mathrm{d}x} - \frac{\omega_m}{\omega_m^2 + \Delta_b^2}\frac{\mathrm{d}|\Delta(x)|}{\mathrm{d}x} = \frac{1}{\tau v_x}C_{a1}$$
$$\frac{\mathrm{d}C_{a1}}{\mathrm{d}x} = \frac{1}{\tau v_x}\big(C_{s2} - \langle C_{s2}\rangle\big) \tag{19}$$

and

$$\frac{\mathrm{d}C_{s1}}{\mathrm{d}x} - \frac{\Delta_b}{\sqrt{\omega_m^2 + \Delta_b^2}}\frac{\mathrm{d}\,\delta\chi(x)}{\mathrm{d}x} = -\frac{1}{\tau v_x}C_{a2}$$
$$\frac{\mathrm{d}C_{a2}}{\mathrm{d}x} = -\frac{1}{\tau v_x}\big(C_{s1} - \langle C_{s1}\rangle\big). \tag{20}$$

Here $\Delta_b(T)$ stands for the modulus of the pairing potential in the bulk of the superconductor. We are neglecting corrections to the bulk value $\Delta_b(T)$ due to the current flow, because they have the order D^2. The functions $C_{s1,2}$, $C_{a1,2}$ depend on x, ω_m, and $\mu = \cos\theta$ (where θ is the angle between the direction of the momentum and the x-axis). C_{s2}, C_{a2} are even functions of x; C_{s1}, C_{a1} are odd functions of x. So we shall consider for definiteness the region $x \geqslant 0$. The boundary conditions to these systems at $x = 0^+$ result from equation (13):

$$C_{a1} = -\frac{D(\mu)\Delta_b\omega_m}{\omega_m^2 + \Delta_b^2}\sin^2\frac{\chi_0}{2} \qquad C_{a2} = \frac{D(\mu)\Delta_b \sin\chi_0}{2\sqrt{\omega_m^2 + \Delta_b^2}}. \tag{21}$$

From equations (19), (20), and boundary conditions (21), we see that the ω_m-dependence factorizes according to

$$C_{a1(s2)} = \frac{\Delta_b\omega_m}{\omega_m^2 + \Delta_b^2}F_{a(s)}(x, \mu)$$
$$C_{a2(s1)} = \frac{\Delta_b}{\sqrt{\omega_m^2 + \Delta_b^2}}G_{a(s)}(x, \mu). \tag{22}$$

Substituting this factorization into the self-consistency condition for the pairing potential (5), we get

$$\delta\Delta(x)\sum_{\omega_m>0}\frac{\Delta_b^2}{(\omega_m^2 + \Delta_b^2)^{3/2}} = -\langle F_s\rangle\sum_{\omega_m>0}\frac{\Delta_b\omega_m^2}{(\omega_m^2 + \Delta_b^2)^{3/2}}$$
$$\langle G_s\rangle = 0. \tag{23}$$

The angular averaging means

$$\langle F_s \rangle(x) = \int_0^1 \mathrm{d}\mu\ F_s(x,\mu)$$

and similarly for $\langle G_s \rangle(x)$. One should not forget that there is a cut-off frequency in the summation over the Matsubara frequencies and

$$1 = 2\pi T\lambda \sum_{\omega_m > 0} \frac{1}{\sqrt{\omega_m^2 + \Delta_b^2}}. \tag{24}$$

Using the expressions (23) we can find $F_s(x,\mu), G_s(x,\mu)$ from equations (19), (20), and boundary conditions (21). Substituting them back into (23), we get the equations

$$\int_0^\infty \mathrm{d}u\ \frac{|\Delta|'(u)}{\Delta_b}(1+\kappa(T)) \int_0^1 \mathrm{d}\mu\ [\mathrm{sgn}(x-u)\mathrm{e}^{-|x-u|/l\mu} - \mathrm{e}^{-(x+u)/l\mu}]$$
$$+ 2\sin^2\frac{\chi_0}{2}\int_0^1 \mathrm{d}\mu\ D(\mu)\mathrm{e}^{-x/l\mu} = 0 \tag{25}$$

and

$$\int_0^\infty \mathrm{d}u\ \delta\chi'(u) \int_0^1 \mathrm{d}\mu\ [\mathrm{sgn}(x-u)\mathrm{e}^{-|x-u|/l\mu} - \mathrm{e}^{-(x+u)/l\mu}]$$
$$+\ \sin\chi_0 \int_0^1 \mathrm{d}\mu\ D(\mu)\mathrm{e}^{-x/l\mu} = 0. \tag{26}$$

The temperature-dependent $\kappa(T)$ in equation (25) is given by

$$\kappa(T) = \left(\Delta_b^2 \sum_{\omega_m} (\omega_m^2 + \Delta_b^2)^{-3/2}\right) \Big/ \left(\sum_{\omega_m} \omega_m^2 (\omega_m^2 + \Delta_b^2)^{-3/2}\right). \tag{27}$$

This has the smallness of λ (the superconducting coupling constant) and can be omitted from equation (25). If we integrate equation (26) over x from 0 to x_0 we arrive at

$$\int_0^\infty \mathrm{d}u\ \delta\chi'(u) \int_0^1 \mu\ \mathrm{d}\mu\ [\mathrm{e}^{-|x_0-u|/l\mu} - \mathrm{e}^{-(x_0+u)/l\mu}] + \sin\chi_0 \int_0^1 \mathrm{d}\mu\ \mu D(\mu)\mathrm{e}^{-x_0/l\mu}$$
$$= \sin\chi_0 \int_0^1 \mathrm{d}\mu\ \mu D(\mu). \tag{28}$$

From this equation we get that for large x

$$2l\,\delta\chi'(\infty)/3 = \sin\chi_0 \int_0^1 \mathrm{d}\mu\ \mu D(\mu). \tag{29}$$

Equation (29) means that in the first order in transparency the Josephson current

$$j = \frac{\pi}{2} e v_F N(0) \sin\chi_0\, \Delta_b \tanh\frac{\Delta_b}{2T} \int_0^1 \mathrm{d}\mu\ \mu D(\mu) \tag{30}$$

is equal to the bulk supercurrent

$$j_S = \frac{\pi}{3} e N(0) v_F l\, \delta\chi'(\infty) \Delta_b \tanh\frac{\Delta_b}{2T}. \tag{31}$$

So equation (28) describes conversion of the Josephson current into the bulk supercurrent in the first order in transparency.

Equation (25) analogously results in the relationship for large $x \gg l$:

$$\frac{|\Delta|'}{\Delta_b} = \frac{3}{l}\sin^2\frac{\chi_0}{2}\int_0^1 \mathrm{d}\mu\ \mu D(\mu). \tag{32}$$

The results thus obtained are valid within distances of the order of l from the junction because of the approximation employed, $M_i = \langle b_{si}\rangle/\tau v_x$. We can use the Usadel equation for the description of the behaviour of the superconducting order parameter at distances $x \gg l$ from the junction. This equation provides the characteristic scale $\xi(T)$ of the change of Δ. Hence equation (32) can be regarded as the boundary condition to the equation for $|\Delta|(x)$ on this larger scale. We recall that the Usadel equation can be written as

$$\omega_m f - \frac{v_F l}{6}\left(g\,\nabla^2 f - f\,\nabla^2 g\right) = \Delta g \tag{33}$$

where functions f, g depend only on $\boldsymbol{R}, \omega_m$ and imply Green's functions averaged over the Fermi surface. Using equation (33) and the asymptotic solutions of equations (19), (20) we arrive at a linear equation for small $\delta|\Delta|(x) = |\Delta(x)| - \Delta_b$:

$$\begin{aligned}\sum_{\omega_m>0} \frac{\omega_m^2}{(\omega_m^2+\Delta_b^2)^{3/2}}\Bigg[\frac{\xi_m}{2}\int_0^\infty \mathrm{d}u\ \delta|\Delta|''(u)\mathrm{e}^{-|x-u|/\xi_m} \\ -\frac{\xi_m}{2}\int_0^\infty \mathrm{d}u\ \delta|\Delta|''(u)\mathrm{e}^{-(x+u)/\xi_m} + \kappa(T)\,\delta|\Delta|(0)\mathrm{e}^{-x/\xi_m}\Bigg] \\ = \delta|\Delta|(x)\sum_{\omega_m>0}\frac{\Delta_b^2}{(\omega_m^2+\Delta_b^2)^{3/2}}.\end{aligned} \tag{34}$$

Here, the following notation was introduced:

$$\xi_m = \frac{\sqrt{v_F l/6}}{(\omega_m^2+\Delta_b^2)^{1/4}}. \tag{35}$$

The solution of equation (34) can be approximated by

$$\begin{aligned}\delta|\Delta| &= -K\mathrm{e}^{-x/b(T)} \\ b(T) &\sim \xi(T) = \sqrt{\frac{v_F l}{\Delta_0}}\left(1-\frac{T}{T_c}\right)^{-1/2}\end{aligned} \tag{36}$$

where K stands for a constant and Δ_0 is the modulus of the bulk pairing potential at $T = 0$. The approximation for the $|\Delta|(x)$ variation becomes exact for temperatures close to the critical temperature T_c. If we integrate equation (34) over x from 0 to ∞ we shall arrive at the equation for $b(T)$:

$$\sum_{\omega_m>0}\frac{\omega_m^2}{(\omega_m^2+\Delta_b^2)^{3/2}}\frac{\xi_m^2}{b+\xi_m} = b\sum_{\omega_m>0}\frac{\Delta_b^2}{(\omega_m^2+\Delta_b^2)^{3/2}}. \tag{37}$$

The function $b(T)$ resulting from this equation is plotted in figure 1. Near T_c we have an exact relationship:

$$b(T) = \frac{\pi}{4\sqrt{3\gamma}}\xi(T) \tag{38}$$

where $\gamma = 1.78$ is Euler's constant. Using the boundary condition (32), we find the value of K in equation (36) and arrive at

$$\delta|\Delta| = -\frac{3b(T)}{l}\Delta_b\sin^2\frac{\chi_0}{2}\mathrm{e}^{-x/b(T)}\int_0^1 \mathrm{d}\mu\ \mu D(\mu). \tag{39}$$

The pairing potential suppression near the junction for temperatures close to the critical one was noted in reference [6] (see equation (33.9) there).

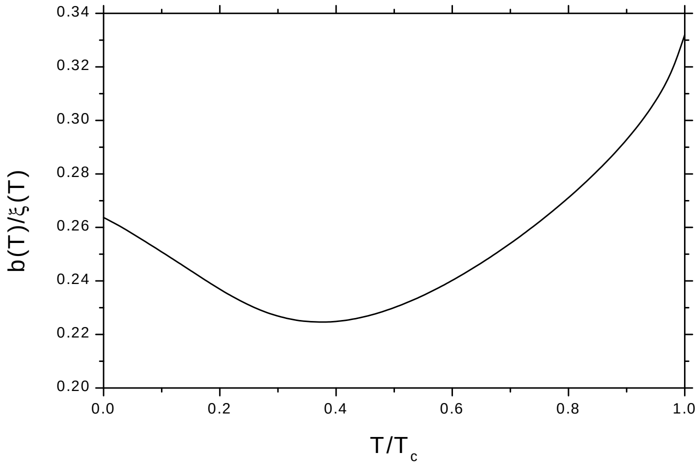

Figure 1. The $b(T)/\xi(T)$ ratio resulting from equation (37).

Because of the large ratio $b(T)/l$, the main contribution to $\boldsymbol{b}_a$ in the second order in transparency comes from the terms in equation (13) which involve the change of the modulus of the superconducting order parameter. Thus we obtain the Josephson current up to the second order in D:

$$j = \frac{\pi}{2} e v_F N(0) \sin\chi_0\, \Delta_b \int_0^1 \mathrm{d}\mu\; \mu D(\mu) \times \left[\tanh\frac{\Delta_b}{2T} - \frac{3b}{l}\sin^2\frac{\chi_0}{2}\left(\tanh\frac{\Delta_b}{2T} + \frac{\Delta_b}{2T}\cosh^{-2}\frac{\Delta_b}{2T}\right)\int_0^1 \mathrm{d}\mu\; \mu D(\mu)\right]. \tag{40}$$

Since $2\sin^2(\chi_0/2) = 1 - \cos\chi_0$, the expression for the Josephson current involves a harmonic $\sin 2\chi_0$ term. It is worth noting that current–phase relationship measurements at a junction with controllable transparency are within the capabilities of modern experimental techniques [7].

5. Discussion

In this paper we derived an expression for the Josephson current (40) up to the second order in transparency of the junction. The effect of the pairing potential suppression near the interface turned out to be essential for the second-order correction. An approximation for the M-term in (9) allowed us to find the spatial behaviour of the quasiclassical Green's function within distances of the order of l from the junction. Thus we obtained the boundary condition (32) for the spatial behaviour of the pairing potential on a larger scale $\xi(T)$. One can use the Usadel equation for the description of the behaviour of the pairing potential on this scale, which leads to equation (34). The result (40) is valid as long as $D\xi(T)/l \ll 1$ (the approximation (36) is not valid otherwise). Note the difference between the plane-junction situation discussed in this paper and the case of the point contact (see e.g. reference [3]). In the latter case the modulus of the pairing potential in the Eilenberger equations can be taken as constant. Contrary to the case for the point contact, the correction (40) results in the current having its maximum value for the phase jump χ_0 smaller than $\pi/2$; see the schematic plot in figure 2. Besides, this correction is more pronounced for the temperatures close to the critical one. These features were observed in the experiment of Koops *et al* [7].

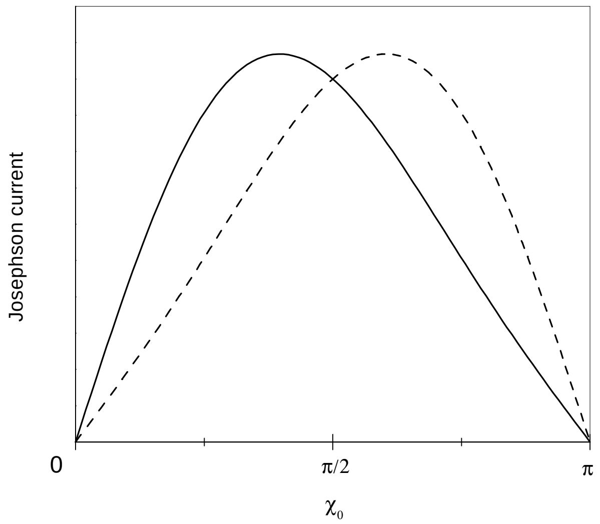

Figure 2. The schematic plot of the current–phase relationship in the case of low transparency for plane contact (solid line) and point contact (dashed line).

The second harmonic in the current–phase relationship was also obtained in [5]. It was shown to result from those terms in the effective boundary condition (61) which were quartic in quasiclassical propagators. The difference of our result, for the amplitude of the second harmonic, from that of Lambert *et al* is caused by the variation of the superconducting order parameter near the interface on the scale $\xi(T)$ which is much larger than l. Thus one cannot substitute the bulk values of the quasiclassical propagators into the effective boundary condition as was done in reference [5] (see their expressions for G, F following equation (70)).

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